Exercise 30

(a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$ and C is given by $r(t) = 2t\,\mathbf{i} + 3t\,\mathbf{j} - t^2\,\mathbf{k}$, $-1 \leq t \leq 1$.

(b) Illustrate part (a) by using a computer to graph C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm\frac{1}{2}$ (as in Figure 13).

Solution

With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_{-1}^{1} \langle x(t), -z(t), y(t)\rangle \cdot \frac{d}{dt}\langle 2t, 3t, -t^2\rangle\,dt \\
&= \int_{-1}^{1} \left\langle 2t, t^2, 3t\right\rangle \cdot \langle 2, 3, -2t\rangle\,dt \\
&= \int_{-1}^{1} [(2t)(2) + (t^2)(3) + (3t)(-2t)]\,dt \\
&= \int_{-1}^{1} (4t - 3t^2)\,dt \\
&= (2t^2 - t^3)\Big|_{-1}^{1} \\
&= 2[1^2 - (-1)^2] - [1^3 - (-1)^3] \\
&= -2.
\end{aligned}$$

Below is a plot of the vectors from the vector field corresponding to $t = \pm 1$ and $t = \pm \frac{1}{2}$.