Pedagogical Research
2024, 9(2), em0198
e-ISSN: 2468-4929
https://www.pedagogicalresearch.com

MODESTUM

Research Article

OPEN ACCESS

Parabola or second-degree polynomial function exploration in initial teacher training: Integrating intuition and didactic situations

Renata Teófilo de Sousa [1*], Francisco Régis Vieira Alves [2], Helena Maria Barros de Campos [3,4]

[1] Secretary of Education of Ceará, Sobral, BRAZIL
[2] Federal Institute of Science, Education and Technology of Ceará, Sobral, BRAZIL
[3] University of Trás-os-Montes and Alto Douro, Vila Real, PORTUGAL
[4] Research Centre on Didactics and Technology in the Education of Trainers-CIDTFF, University of Aveiro, PORTUGAL
***Corresponding Author:** rtsnaty@gmail.com

Citation: de Sousa, R. T., Alves, F. R. V., & de Campos, H. M. B. (2024). Parabola or second-degree polynomial function exploration in initial teacher training: Integrating intuition and didactic situations. *Pedagogical Research, 9*(2), em0198. https://doi.org/10.29333/pr/14340

ARTICLE INFO

Received: 27 Sep. 2023

Accepted: 17 Feb. 2024

ABSTRACT

This work is the result of a master's investigation in Brazil, which discusses the teaching of the parabola in the initial training of mathematics teachers. Our theoretical framework addresses the relationship between intuition and the dialectics of the theory of didactic situations, which supports the analysis of the results of this study Its objective was to identify and register categories of intuitive reasoning manifested by the subjects when solving a didactic situation involving the parabola with GeoGebra software contribution. The methodology adopted was didactic engineering, experimented with eight students in initial training at a Brazilian public university, among 6^{th} and 9^{th} undergraduate semesters. The posterior analysis and validation allowed us to verify the need to discuss the parabola, articulating its geometric, algebraic, and analytical views, as well as to reinforce the importance of its teaching with the use of technology.

Keywords: parabolas, GeoGebra, theory of didactic situations, intuitive reasoning

INTRODUCTION

Parabolas are of fundamental importance for the development of areas of knowledge such as architecture, engineering, and physics with applications in everyday life such as civil construction, satellite dishes, solar ovens, mirrors, and car headlights, among others. However, in its approach in basic education and extension of its study in higher education, there is little discussion about its applicability, especially from the perspective of analytical geometry (Cerqueira, 2015; Macedo, 2015) or even its association with the use of technology.

This is a topic that is not prioritized in the Brazilian curriculum, "in fact, it is not a topic that appears frequently in selection exams in various parts of the country" (Siqueira, 2016, p. 6), which causes difficulties for the student when studying higher level topics such as analytical geometry and differential and integral calculus, for example. Starting from this problem and considering the importance of discussing the topic, we rely on the French aspect of mathematics didactics, given the fact that its studies bring together different currents that question the canonical paradigms that permeate the training of mathematics teachers and the teaching of its curricular components (Alves, 2017).

In the meantime, we bring as a theoretical contribution the theory of didactic situations (TDS) (Brousseau, 2002) as a guide for the experiment, as well as didactic engineering (DE) (Artigue, 2020) as a research methodology. Associated with these theories, we associate intuition and its categorization at different levels, as proposed in Fischbein (1987), in what he calls categories of intuitive reasoning, aiming to contribute to the understanding of how the construction of mathematical thinking occurs based on this ontological faculty. As a way of identifying the manifestation of these categories, we also count on the technological support of the GeoGebra software.

Given the above, the objective of this work is to identify and record intuitive reasoning categories expressed by students in initial training, based on their actions and strategies to solve a didactic situation involving the parable with input from GeoGebra.

To this end, we structured this research on the assumptions and phases of DE. DE is a research methodology that, according to Artigue (2020), can be described by an experimental scheme based on didactic achievements within the classroom, that is, in the design, implementation, observation and analysis of teaching sessions. In this investigation we developed its four phases–(a) preliminary analyses, (b) a priori design and analysis, (c) experimentation, and (d) a posteriori analysis and validation–observing and improving a DE aimed at teaching parables, with a view to contributing to the development of future mathematics teachers.

The experiment was carried out at a Brazilian public university, with eight students in initial training, between the 6th and 9th semesters of the mathematics degree course. Data collection occurred through photographic records, audios, videos, written material, and files created in the GeoGebra software. The didactic situation was structured based on TDS and the data were analyzed based on the categories of intuitive reasoning, being organized according to the assumptions of DE.

The study of the parabola plays a significant role in the mathematics teacher training, as this concept is crucial for the development of algebraic and geometric understanding among future educators. The relevance of this topic is intrinsically linked to its applicability in various mathematical areas and its importance in solving real-world problems. In this context, we emphasize the significance of the parabola in teacher education, the challenges commonly encountered by students in this subject, and the relevance of the study of the parabola within the field of mathematics education.

Therefore, in the following sections we present the development of the engineering phases, which deal with the theoretical contribution, the structuring and development of the experiment and analysis of the results, as well as the authors' considerations.

THEORETICAL FRAMEWORK

Theory of Didactic Situations, Categories of Intuitive Reasoning, & Their Association

TDS brings a theoretical model that aims to understand the dialectical relationship established between the main actors in a didactic system–the teacher, the student and knowledge–as well as the environment (*milieu*) in which the situation of a specific didactic situation develops. Based on this, TDS aims to encourage the student to behave as a researcher, where, based on a set of dialectics, the student can develop and be able to formulate hypotheses and concepts, while the teacher provides favorable situations so that he transforms the information into knowledge for himself.

Brousseau (2002) explains that student learning derives from their adaptation to a milieu intertwined with contradictions, difficulties, and imbalances. The knowledge resulting from this adaptation manifests itself through new responses, which in turn provide evidence of learning. Thus, we understand that student autonomy is developed through decision-making, reflection and organization of ideas based on their prior knowledge, as long as the milieu is designed by the teacher in order to produce such imbalances and their consequent search for understanding and apprehension of knowledge.

TDS organizes the student's learning process based on situations or dialectics, called action, formulation, validation, and institutionalization, where the first three dialectics are considered the didactic situation, which is designed so that the student interacts with an environment without the teacher's intervention. For the development of this work, we were interested in the path of mathematical reasoning in the development of TDS dialectics.

To build a model of a subject's mathematical reasoning based on the notion of situation, it is necessary to understand that reasoning concerns a domain that is not restricted to formal, logical or mathematical structures, despite being made up of an ordered set of statements linked, combined or opposed to each other, respecting certain restrictions that can be made explicit in the solution of a problem (Brousseau & Gibel, 2005).

A reasoning can be characterized by the role it plays in a situation, that is, by its function in that situation. Thus, such a function can be deciding about something, informing, convincing, or explaining (Brousseau, 2002). From this perspective, the function of reasoning varies according to the type of situation in which it occurs, having a direct relationship with the dialectical movement of TDS, that is, whether it is a situation of action, formulation, or validation. Thus, Brousseau and Gibel (2005) seek to distinguish levels of mathematical reasoning, considered more or less degenerate, and which adapt to different types of situations in TDS, as summarized below:

1. **1st level reasoning (L1):** It can be characterized by a type of reasoning that is not formulated as such, however it can be attributed to the subject based on their actions, and constructed as a model of this action, being considered as an implicit model relating to the action situation in TSD.
2. **2nd level reasoning (L2):** It can be considered as incomplete reasoning from a formal point of view, but with gaps that can be, implicitly, filled by the student's actions in a situation in which a complete formulation would not be justified. This type of reasoning appears in situations, where communication is necessary, being related to the formulation phase.
3. **3rd level 3 reasoning (L3):** It can be defined as formal, global, and concluded reasoning, based on a set of correctly related inferences, which make clear mention of the elements of the situation or knowledge considered to be shared by the class, even if not yet it is postulated that such reasoning is absolutely correct. Reasoning at this level is characteristic of validation situations.

We consider that each stage of reasoning is incorporated into logical and mathematical justifications considered standard, and its validity and relevance appear to be autonomous. In the authors' proposal, the interpretation of students' solutions must consider a larger and more complex system, if the teacher's intention is to instigate them or even explain why such forms of reasoning, correct or not, were produced.

In parallel, with regard to intuition, focused strictly on mathematics, this has been the subject of discussions in the field of mathematics education. We can say that intuition refers to a product of representations made from reality and, in this sense, has an auxiliary role in the students' learning process, which can be considered by the teacher. Here we bring intuition in an articulated way to the construction of mathematical reasoning, also discussed from a categorization into levels from the perspective of the Romanian psychologist and researcher Fischbein (1987).

Regarding the categorization of intuition, Fischbein (1987) seeks to articulate its different types and their relationship with problem solutions, classifying them into what he calls categories of intuitive reasoning: affirmative, conjectural, anticipatory, and conclusive intuitions. We bring a brief description of these, from the author's perspective:

1. **Affirmative intuitions:** They are representations, interpretations or understandings directly accepted by human beings as natural truths, evidently and intrinsically significant.
2. **Conjectural intuitions:** In this model of intuition there is an explicit approach to solving a problem, however, the subject is not involved in an effort to solve it. In other words, this type of intuition refers to assumptions linked to the feeling of certainty.
3. **Anticipatory intuitions:** This type of intuition provides an absolute point of view, preceding the solution of a problem, which precedes the fully developed analytical resolution. The subject sees all the steps to their solution and understands the path to follow to achieve the expected answer.
4. **Conclusive intuitions:** They synthesize a globalized and structured view of the basic ideas for solving a problem, previously elaborated, thus depending on the other three types of intuition mentioned above. It enables the generalization of the mathematical structure for the proposed problem and replication of the solution model in similar situations.

Fischbein (1987) thoroughly examines the teaching and learning process by considering that, recurrently, the student faces obstacles in their learning, understanding and problem solving at more advanced levels, given the fact that, sometimes, their techniques and strategies of reasoning are driven by implicit and/or intuitive, sometimes inadequate, models. In this sense, the teacher supposedly has the task of investigating and recognizing such models, supporting the student in improving their mental schemes, so that their reasoning is constructed appropriately.

Given the above, we can infer a relationship between what Brousseau and Gibel (2005) propose about the different levels of mathematical reasoning in the development of TDS and what Fischbein (1987) explains in his categorization of intuition. In this sense, we propose a correlation between the authors' ideas, as shown in **Figure 1**.

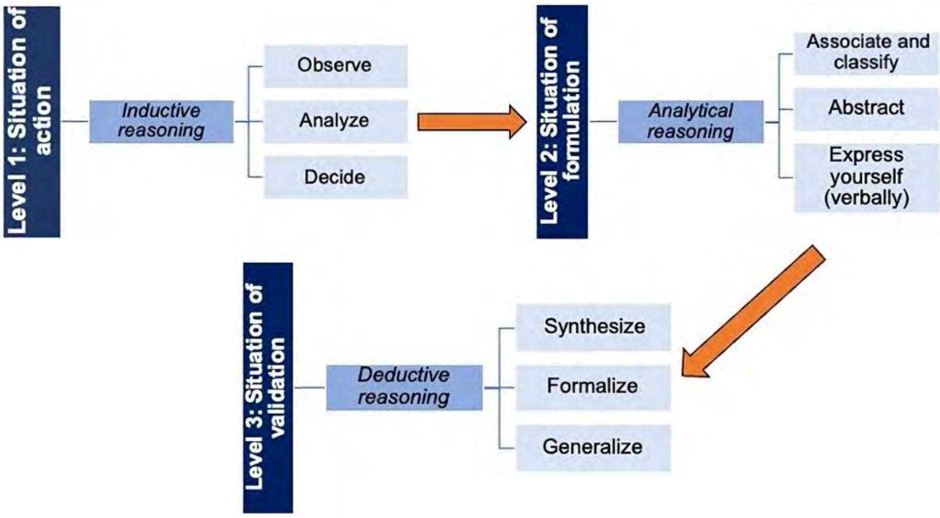

Figure 1. Relationship between levels of reasoning in TDS & intuitive categories (Source: Authors' own elaboration)

When Brousseau and Gibel (2005) propose the 1st level reasoning (L1), describing it as a reasoning model not yet formulated, but related to the subject's position in an action situation in TDS, we can see a similarity with the category Fischbein's (1987) affirmative intuition. This relationship can be seen when Fischbein (1987) proposes that the subject has a preliminary view of the problem and a superficial view of its solution path, being able to observe and analyze, based on structures of inductive thought. At this level of reasoning and intuitive category, the student has not yet taken any action to solve it but is in the process of predicting his hypotheses and then establishing a path that makes sense to him.

The 2nd level reasoning (L2) proposed by Brousseau and Gibel (2005) is considered unfinished from a formal point of view, but with gaps that can be implicitly filled based on the student's performance in a formulation situation in TDS. This reasoning model can be related to the conjectural intuitions proposed by Fischbein (1987) in which the student starts with analytical reasoning about each part of the problem. Thus, the student begins their deductions from a starting point, being able to associate, classify and express themselves verbally, formulating ideas and establishing a path to the solution in a more explicit way.

From this perspective, we can understand that 3rd level reasoning (L3), defined by Brousseau and Gibel (2005) as a formal, global, and finished model, which is based on the sequential connection of inferences articulated in a cohesive way (although such reasoning is not absolutely correct), as a format presented in validation situations in the TDS. It is possible to relate 3rd level reasoning to what Fischbein (1987) proposes as anticipatory intuition and/or conclusive intuition, depending on how this reasoning was produced by the student. We understand that both the levels of reasoning proposed by Brousseau and Gibel (2005) and the categories established by Fischbein (1987) show that the learning trajectory of new reasoning occurs when it is promoted from a single particular means of solving a problem. problem to a universal means of solving all problems of a given type and is integrated as such with the subject's knowledge. In an autonomous situation, reasoning is based on induction, but this induction is supported by a chain of inferences that can be made explicit. Thus, the identification of intuitive models and levels of reasoning requires an a priori theoretical analysis of behaviors, difficulties and procedures that may arise in the class phases and in the development of a didactic situation.

Preliminary Analysis

The preliminary analysis of a study considers the general didactic theoretical framework regarding a given mathematical object, being composed of an investigation that considers the epistemological, historical, and didactic prism (Almouloud & da Silva, 2012). The current teaching of a given mathematical object, its effects, conceptions, and obstacles faced by students must be considered, as well as an analysis of the field, where teaching achievements will be effectively developed (Artigue, 2020).

In this preliminary analysis we present a mathematical discussion about two specific ways of representing the parabola: from the point of view of quadratic functions and as a locus in analytical geometry. We also seek to present some particularities in their teaching and the possible gaps present in initial training regarding the topic.

Parabola & some particularities in its teaching

The parabola is a conic generated by the section of a plane α over a right circular cone. In other words, the parabola comes from the intersection of a second-degree conical surface and a plane parallel to the generatrix of the cone (Venturi, 2003). In Brazilian school books, the definition of the parabola is commonly presented from the perspective of analytical geometry.

One of these definitions appears in Lima (2014, p. 115), where we have that "let *d* be a straight line and *F* be a point outside it. In the plane determined by *d* and *F*, the set of points equidistant from *d* and *F* are called parabola of focus *F* and directrix *d*", which can be represented by the scheme given in **Figure 2**.

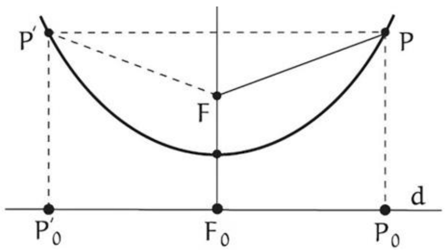

Figure 2. Geometric representation of parabola (Lima, 2014)

According to Lima (2014), point *P* belongs to the parabola of focus *F* and directrix *d*, as the distance *D* from point *P* to *F* is the same distance between point *P* and P_0, where P_0 is a point belonging to directrix *d*. In other words, $D(P, F)=D(P, P_0)$, with the segment PP_0 perpendicular to the directrix *d* and the perpendicular FF_0 lowered from the focus onto the directrix, is configured on an axis of symmetry, being the definition of the parabola as a place geometric. The definitions presented by Lima (2014) and Venturi (2003) are seen in other analytical geometry books, with few differences, with an approach more focused on algebra.

In the context of 2nd degree polynomial functions, the definition of a parabola is presented in textbooks as the graph of the quadratic function $f(x) = a \cdot x^2 + b \cdot x + c$, with $a \neq 0$. This definition is usually accompanied by an explanation that points out that when parabolas represent a quadratic function, they can have their opening (concavity) facing up or down (Leonardo, 2016) without often mentioning why this occurs, or even the difference between a parabola and a catenary.

On several occasions, textbooks have brought examples of suspension bridges as parabolas–in this case the approximation of these, in a similar way to the approximations made in Engineering–as well as the trajectories of projectiles or other structures that refer to this curve, but without mentioning that, in Under certain conditions, suspension bridges have their structures supported by the catenary equation (de Sousa et al., 2022).

Possibly this apparent confusion may occur geometrically since the algebraic expressions of the two objects are different (Lima, 2001). However, we emphasize that it would be an important fact to point out, as it is not just a mere mistake to confuse these two curves, but something that can compromise entire architectural structures.

One way to rewrite the 2nd degree polynomial function, associating it with the parabola equation in analytical geometry, is its canonical form in which the function can be written based on the coordinates of its vertex. If the function is in canonical form, as in $f(x) = a \cdot (x - h)^2 + k$, the vertex is given by the coordinates of the point (h, k). Using a perfect square trinomial, we can rewrite the general form of the 2nd degree function $f(x) = a \cdot x^2 + b \cdot x + c$ to the canonical form: $f(x) = a \cdot (x - h)^2 + k$, with vertex $V\ (x_v, y_v) = (-\frac{b}{2a}, -\frac{\Delta}{4a})$, where $h = -\frac{b}{2a}$, $k = -\frac{\Delta}{4a}$ and $\Delta = b^2 - 4ac$, which can be obtained through elementary calculations.

The canonical form is viable in solving situations in which the vertex coordinates are known. This format is little explored in mathematics classes at school, and many textbooks do not cover this topic (Cerqueira, 2015; Macedo, 2015). Lucena and Gitirana (2016, p. 25) state that the study of the parabola occurs through "teaching that prioritizes algebraic treatment and leaves the geometric interpretation of the represented mathematical object to be desired, which makes it difficult to deepen the concepts in focus". This makes us reflect on the importance of seeking different approaches to teaching this topic in the classroom.

Furthermore, other research such as Bohrer and da Silva Tinti (2021) and Feltes and Puhl (2016) state that the teachers' methodology in the Brazilian context brings with it some gaps, with traditional classes and little use of technological resources or practical applications. Thus, students tend to develop the abstract thinking part little, presenting a narrow and reductive view of the parabola and the quadratic function itself. There is still a lack of a deeper discussion on the concept of quadratic function in the classroom, as well as the exploration of multiple representations of functions, considering the relationship between the concept and the image, and a consequent exploration of (intuitive) mathematical reasoning, as defended by Fischbein (1987).

Based on the above, we explore the parabola in this work, seeking to understand it from a geometric view and the manipulation of its parameters using the GeoGebra software.

A priori analysis

According to Almouloud and da Silva (2012), balance and organization in preparing an *a priori* analysis are important for the success of a didactic situation, as this stage allows the teacher to control the performance of student activities, enabling identification, in a predictive way, of the observed circumstances. In the case of this work, we bring a didactic situation that seeks, from the parabola equation in its explicit form, to find its main elements, which are the focus (*F*), the vertex (*V*), the parameter (*p*) and the equation of the straight line, starting from the data established in the statement and based on this, sketch your solution in GeoGebra:

Proposed didactic situation: Find the main elements of the parabola $y = 4x^2 - 24x + 37$. Diagram this situation in GeoGebra and present your solution.

This didactic situation does not specify which are the main elements for the construction and existence of the parabola, which can encourage subjects to immediately use their knowledge in 2nd grade functions, due to the equation model presented. However, it is worth highlighting that, mathematically, when we refer to the main elements of a parabola, we are mentioning the focus *F*, the directrix *d*, the parameter *p*, the vertex *V* and the axis of symmetry.

In the *action situation*, we hope that pre-service teachers will seek to outline the data of the problem and outline their solution strategy in advance on paper, based on an initial reading. It may happen that one of the subjects tries to resolve the situation using the Bháskara formula, commonly adopted in similar situations. Even if erroneously, this type of reasoning can be interpreted as an *affirmative intuition*.

If this occurs, this pre-service teacher probably demonstrates a still redundant view of the mathematical concept of parabola linked above all to the graph of a quadratic function. Otherwise, pre-service teachers should possibly start a sketch on paper, transforming the explicit equation given into an equation of the form $(x - x_0)^2 = 2p(y - y_0)$, starting from algebraic procedures that may arise in the midst of *conjectural* and *anticipatory intuitions*.

In the *formulation situation*, we aim for participants to exchange ideas with each other, sketching on paper the structure of the explicit equation of the parabola $y = 4x^2 - 24x + 37$ in the form $(x - x_0)^2 = 2p(y - y_0)$, following a path, which rewrites the equation $y = 4x^2 - 24x + 37$ as the perfect square trinomial $(x - 3)^2 = \frac{1}{4}(y - 1)$.

In the *validation situation*, we hope that pre-service teachers will be able to conjecture, through *anticipatory intuitions*, that the vertex of the parabola is at the point $V(3,1)$, its parameter is equivalent to $p = \frac{1}{16}$ and its focus is the point $F(3, \frac{17}{16})$. Based on the vertex and focus found, pre-service teachers must sketch the directrix equation, establishing that $y - \frac{15}{16} = 0$. If the solution is started in GeoGebra, it is possible to plot it by inserting the explicit equation given $y = 4x^2 - 24x + 37$ into the input field. When observing its graph (**Figure 3**), *affirmative* and *conjectural intuitions* may occur in which the pre-service teacher follows a path in search of the main elements of the parabola.

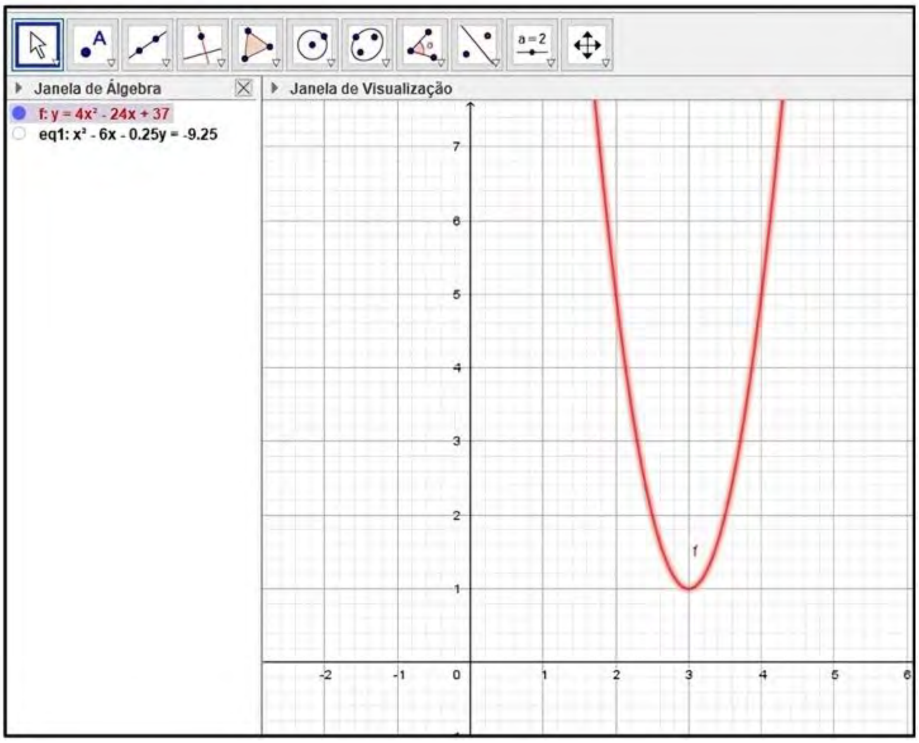

Figure 3. Graph of $y = 4x^2 - 24x + 37$ (Source: Survey data, 2022)

In **Figure 3**, the pre-service teacher can visualize the type of parabola given. Using their prior knowledge to rewrite the equation in its canonical form $(x-3)^2 = \frac{1}{4}(y-1)$, they can enter it in the GeoGebra input field, obtaining (**Figure 4**).

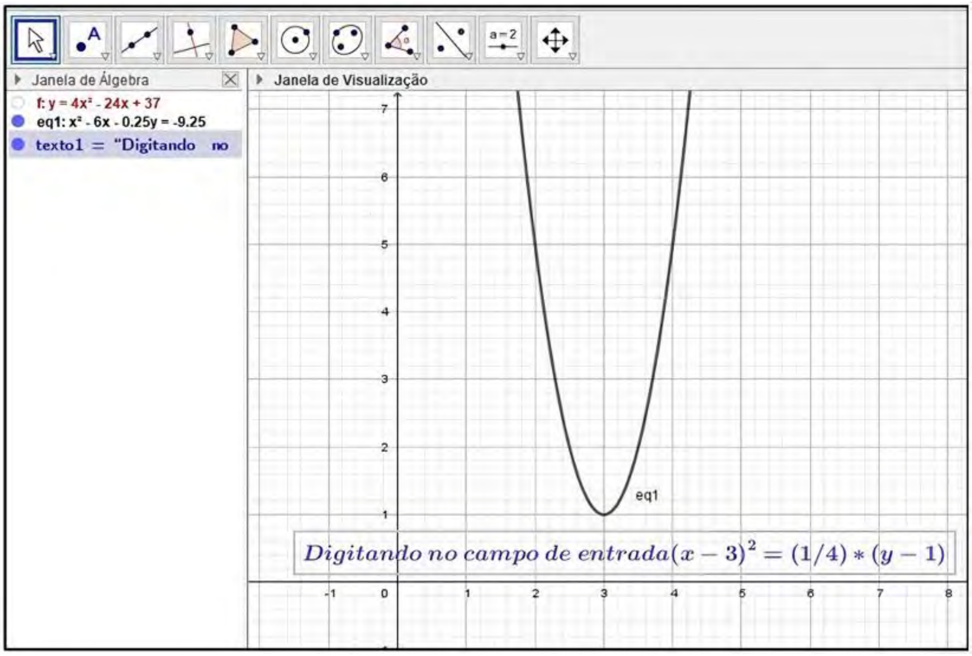

Figure 4. Sketch of $(x-3)^2 = \frac{1}{4}(y-1)$ in GeoGebra (Source: Survey data, 2022)

In **Figure 4**, the pre-service teachers can perceive in the equation, through *conjectural* and *anticipatory* intuitions, the values of the focus, vertex, parameter, and directrix of the parabola, with the help of paper or not. Another possibility would be to use the commands *focus(<conic>), vertex(<conic>)*, and *directrix (<conic>)* relative to *eq1*, finding the solution to the problem (**Figure 5**).

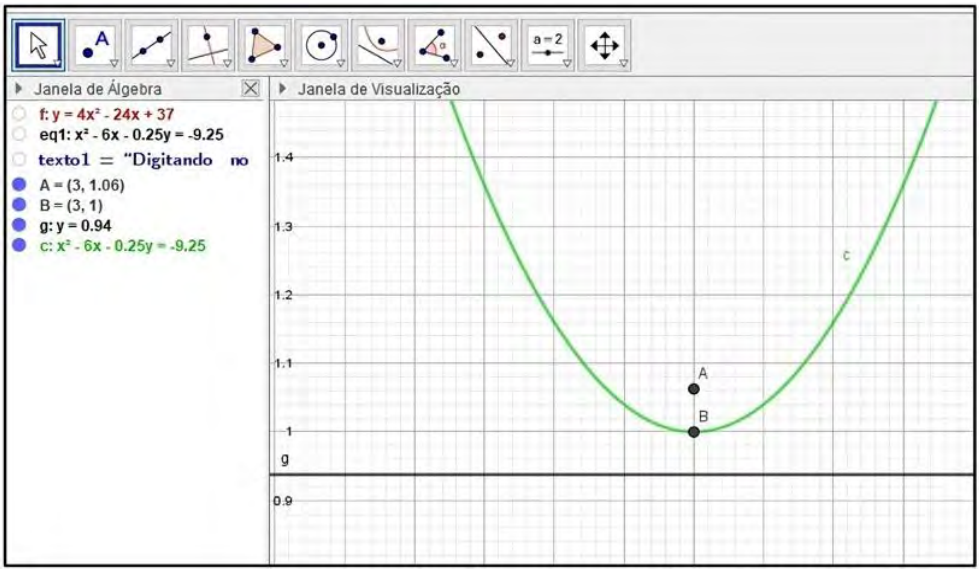

Figure 5. Solving situation in GeoGebra (Source: Survey data, 2022)

In the *institutionalization situation*, the researcher-teacher must intervene by presenting the concept of the explicit equation of the parabola. To do this, we will use a definition based on Lima (2014), which explains that the equation of a parabola with vertex $V(x_0, y_0)$ and axis parallel to the y axis has the standard form: $(x-x_0)^2 = 2p(y-y_0)$. In this case, knowing x_0, y_0, p and explaining y in this equation, we then find an equation presented in its explicit form, in the structure $y = ax^2 + bx + c$, according to the data in the problem statement.

EXPERIMENTATION

The experimentation was developed with eight students in initial teacher training, between the 6th and 9th semesters. As a way of preserving their identities, we named these participants as P_1, P_2, ..., P_8.

Data collection took place through photographic records, audios, computer screen recording videos, written material and files created in the GeoGebra software. The data were analyzed in light categories of the intuitive reasoning and structured based on the assumptions of DE and TDS.

All participants were students from the same undergraduate course and from the same university, in the city of Sobral, Ceará, Brazil. We established criteria to consider subjects eligible for analysis: 100% attendance at training meetings and greater participation/interaction in the proposed activities.

In **Table 1**, we summarize the academic and professional profile of participants.

Table 1. Summary of participants' profile (Survey Data, 2022)

Questions	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
Do you already have an academic degree?	No							
Which semestre are you currently studying?	6th	6th	8th	6th	9th	6th	9th	9th
What subjects involving analytical geometry have you already taken?	- Plane analytical geometry - Vector analytical geometry - Differential & integral calculus I, II, & III							
Do you already work, or have you worked as a teacher? If yes, for how long?	No	No	Yes, 1 year & 6 months	No	No	Yes, 2 months	Yes, 4 years & 8 months	No

In **Table 2**, we summarize the intuitive categories that were recorded from each of the eight participants and in which situation within the adidactic phase of TDS this occurs.

Table 2. Summary of participants' intuitive categories record (Survey Data, 2022)

Participant	Action situation	Formulation situation	Validation situation
P_1	Affirmative and conjectural	Affirmative and conjectural	Not registered
P_2	Not registered	Conjectural and anticipatory	Anticipatory and conclusive
P_3	Conjectural	Conjectural	Not registered
P_4	Not registered	Affirmative	Not registered
P_5	Not registered	Not registered	Not registered
P_6	Affirmative	Conjectural	Conclusive
P_7	Affirmative	Affirmative and conjectural	Anticipatory
P_8	Affirmative	Not registered	Not registered

Next, we describe the results obtained in the development of the didactic situation with these participants, as well as the empirical records of data collection, which provides a better interpretation of the intuitive categories associated with TDS.

Description of On-Site Didactic Situation

This didactic situation presents an equation of the parabola in its explicit form $y = ax^2 + bx + c$ e and asks pre-service teachers to present its main elements. In a standard solution, participants would be expected to find, from the given equation, the coordinates of the focus, the vertex and the directrix equation. However, different solution models have emerged for this.

In the *action situation*, all participants sketched out on paper and pencil the information considered relevant to the solution. However, some of them immediately sought (*affirmative intuition*) to calculate the roots of the given discovery, considering it through the prism of quadratic functions. This was clearly verbalized in the audio recordings of the research, at the time of the action, in a dialogue between the participants and the researcher:

> Can I now highlight the main elements such as the coefficients a, b, and c? (P_7).
>
> What do you actually consider to be the main element of the parabola? (researcher).
>
> The vertex, the roots and a, b, and c (P_7).
>
> Here, it's easy, I'll highlight everything I think about the parabola, and I'll graph it (P_6).
>
> What can you remember about that? (researcher for P_8).
>
> The roots (P_8).
>
> The equation, like this, I do not remember how I find the focus (P_1).
>
> I'm finding the "delta" and I'm going to put the vertex here on the graph, then the roots, then I'll look for other things (P_6).

Note that they immediately presented an *affirmative intuition*, declaring its roots (P_8) and its coefficients (P_6, P_7, and P_8) as the main elements of a parabola, with emphasis on the point of view of the quadratic function. Only P_1 mentioned the focus but said he did not know how to find it. In the written records of subjects P_1, P_6, and P_7, highlighted in **Figure 6**, we also have a demonstration of this point of view.

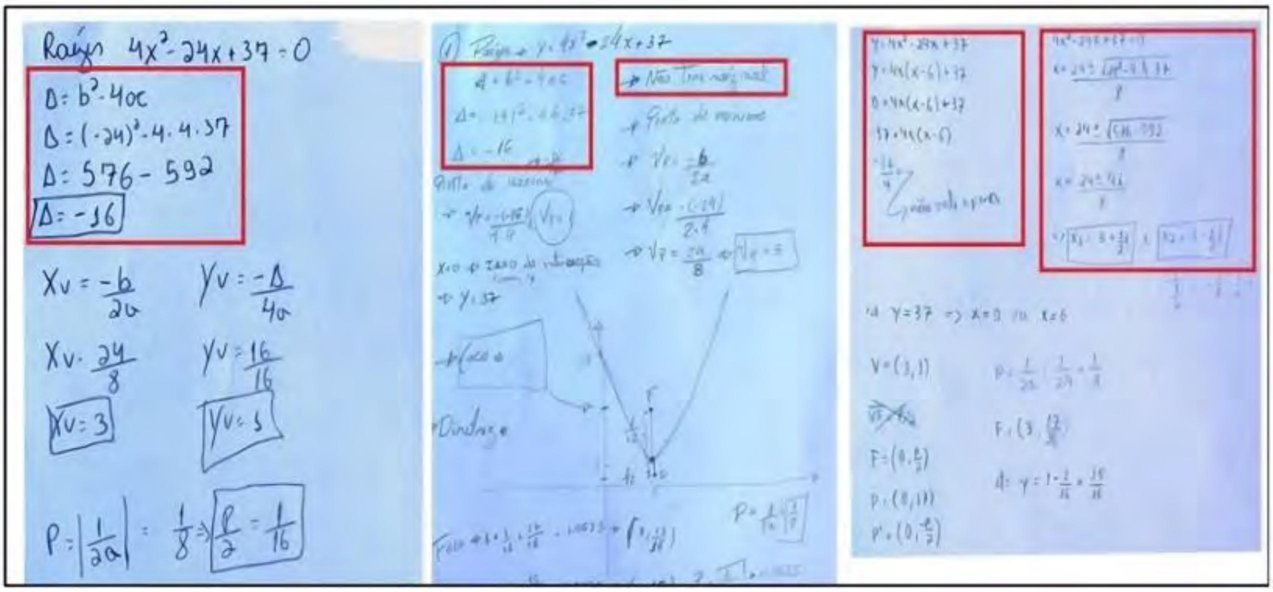

Figure 6. Action situation–P_1, P_6, & P_7, from right to left, respectively (Source: Survey data, 2022)

In **Figure 6**, participants P_6 and P_7, after attempting to calculate the roots (possibly aiming to find exact roots), immediately proceeded to search for the coordinates of the parabola's vertex, still in a manifestation of *affirmative intuitions* arising from their previous knowledge. The other participants looked for the vertex of the parabola using the relations $x_v = -\frac{b}{2a}$ and $y_v = -\frac{\Delta}{4a}$, with the exception of P_3, who in a *conjectural intuition* calculated the derivative of $y = 4x^2 - 24x + 37$, finding $y = 8x - 24$ and, when adopted $y = 0$, obtained $x = 3$. Substituting into the initial equation for $x = 3$, P_3 obtained $y = 1$, which generated the coordinates of the vertex in $V(3, 1)$. We can see this record written in **Figure 7**, in the region highlighted in red.

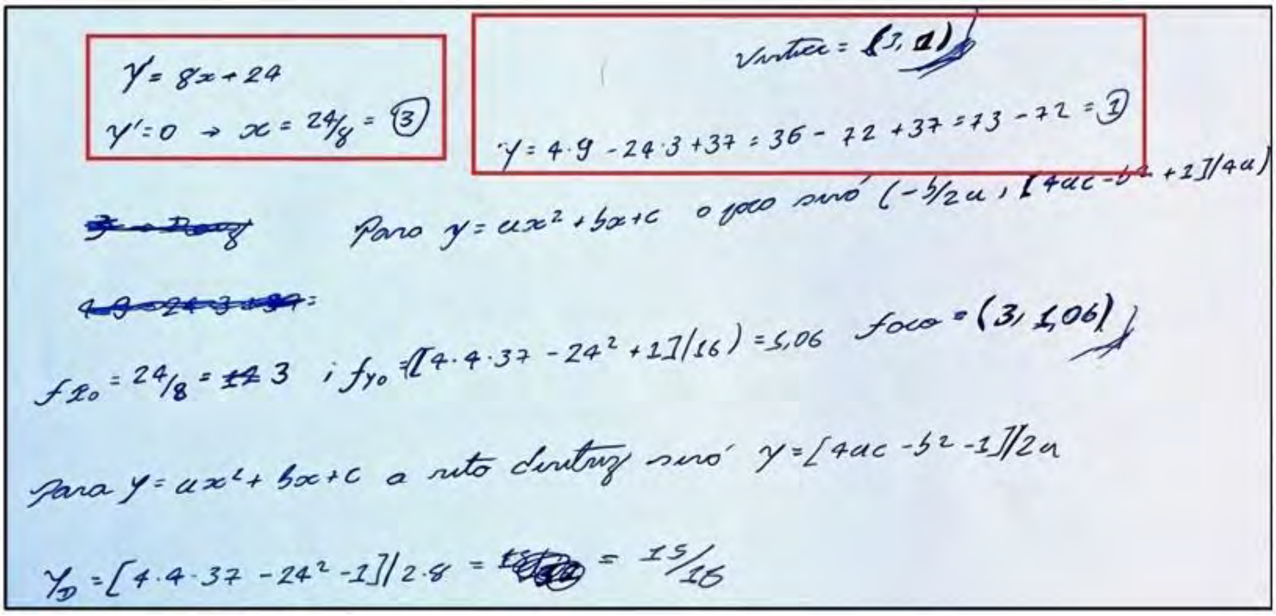

Figure 7. P_3 action situation for calculating vertex (Source: Survey data, 2022)

During the *formulation situation*, the participants switched to the computational environment, entering the information found manually in GeoGebra. To better understand the path that encompasses mathematical reasoning linked to intuition, in the following paragraphs we describe some of the video recordings of the research and constructions carried out in GeoGebra by the participants. Participant P_1 started the formulation situation by typing the function $y = 4x^2 - 24x + 37$ in the GeoGebra input field. Then, he used the "segment" tool and created a segment connecting an arbitrary point that he considered as the focus and the other point on the *x* axis, which were apparently equidistant from what P_1 considered as the vertex of the parabola. Then, he used the "midpoint" tool to discover the midpoint of the segment *AB* and considered it as the vertex (**Figure 8**).

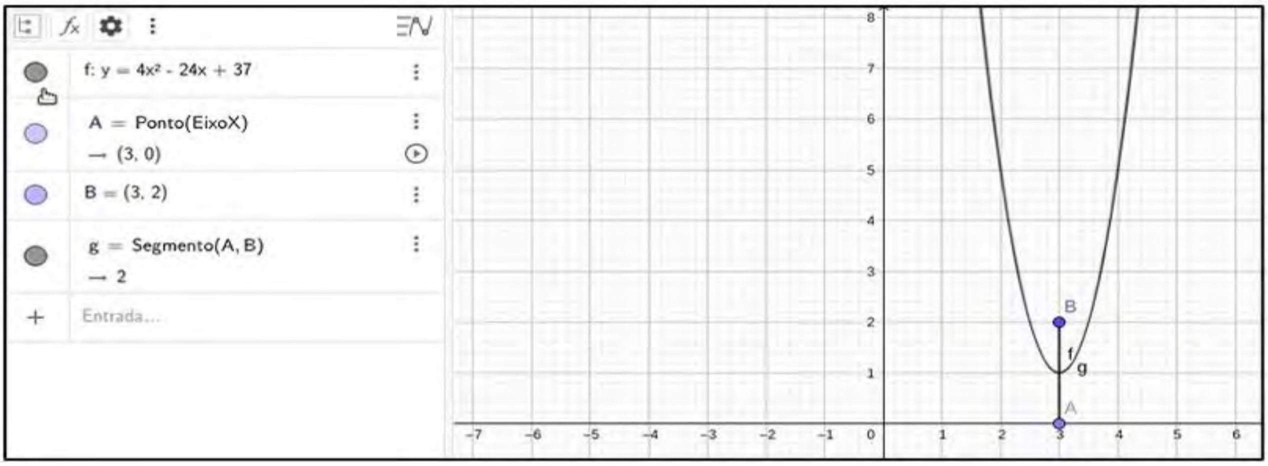

Figure 8. P_1 formulation situation (Source: Survey data, 2022)

We understand that in this arbitrary choice of points for the focus and vertex, P_1 expressed a *conjectural intuition*. However, P_1 disregarded the fact that the distance from the focus *F* to any point *P* on the parabola, as well as the distance from this same point *P* to the straight line must have the same measure, regardless of where the point *P* is located on the curve. The conjectural structure of intuitions is usually covered by their apparent obviousness and certainty. "This is, in fact, the fundamental role of intuitive cognitions: providing certainty to extrapolated ideas" (Fischbein, 1987, p. 53), which made P_1 reflect on his path.

After that, P_1 positioned a point on the coordinates $(0;\ 37)$, considering it as the coefficient *c* of a quadratic function. In this passage we interpret that an *affirmative intuition* occurred, given the prior knowledge about quadratic functions and their coefficients expressed from the beginning by this subject. However, when discussing with other colleagues, P_1 realized that his conjectures were not in fact appropriate. Thus, he began to seek the focus, the vertex, and the directrix of the parabola. Given his difficulties in finding specific commands in GeoGebra, P_1 calculated manually and then built these elements in the software but did not present a final solution.

Participant P_2 also started by entering the function $y = 4x^2 - 24x + 37$ in the GeoGebra input field. He opened all the tool tabs and looked for commands that would help him. Then, he typed the word "focus" in the input field and noticed that GeoGebra provided a series of different commands, depending on the available elements constructed by the user. Here we see the manifestation of a *conjectural intuition* by P_2, when testing the commands and their discoveries. P_2 typed the command *focus(<conic>)* and tried to enter *focus(f[x])*, without success. By rewriting the function as an equation in the input field, in the form $f: y = 4x^2 - 24x + 37$ and using the *focus(<conic>)* command again, the focus coordinates were obtained as $F(3;\ 1.06)$. Likewise, P_2 demonstrated an *anticipatory intuition*–he tried to do the same with other elements, such as the directrix and the vertex, following the same route. Then, he constructed the directrix using the *directrix(<conic>)* command and inserting *directrix(f)*, finding $g: y = 0.94$. Using the same structure, he used the command *vertex(<conic>)*, inserting *vertex(f)* and obtaining $V(3,1)$ and inserting *parameter(<parabola>)* and *parameter(f)*, obtaining the value $p = 0.13$.

To construct the latus rectum, P_2 realized that just by typing the word "axis", GeoGebra opened a series of commands and, from them, selected "*axis(<conic>)*", entering *axis(f)* obtained *latus rectum* and another straight line, perpendicular to *latus rectum* of parabola and passing through *vertex V*. We consider his conjectures successful, and his final construction was (**Figure 9**). Participant P_3, unlike the others, started by entering the coordinates of the vertex $V(3, 0)$. First, P_3 did manual calculations and then used the software just to enter his findings. However, according to the given equation, the vertex ordinate is incorrect. After the interval, P_3 changed the vertex coordinates to $V(3, 1)$. Soon after, P_3 used the "derivative" command and calculated the derivate of the function $f(x) = 4x^2 - 24x + 37$, obtaining $8x - 24$. After that, he calculated $a = -\frac{b}{a} = \frac{24}{8} = 3$. But something that caught our attention was the way he thought about calculating the focus, as we can see in the video clipping, in **Figure 10**.

Figure 9. P_2 formulation situation (Source: Survey data, 2022)

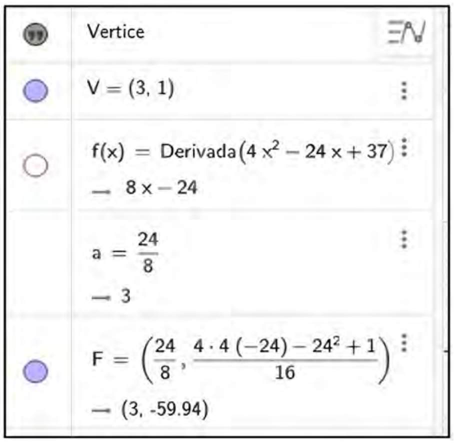

Figure 10. P_3 focus calculation (Source: Survey data, 2022)

P_3 considered the focus as $F = (-\frac{b}{a}; \frac{4ab-b^2+1}{a^2})$, which configures a *conjectural intuition*, originating from the development of what we know as the Bháskara Formula, used for the calculating roots of a quadratic function. The coordinates obtained were $F(3, -59, 94)$, which does not correspond to the correct answer. After that, P_3 inserted the function $g(x) = 4x^2 - 24x + 37$ in the input field and made a small correction in the expression for calculating the focus, obtaining $F = (3;\ 1.06)$, being the correct value. The expression for focus in this case would be $F = (-\frac{b}{2a}; \frac{4ac-b^2+1}{4a})$, where P_3 obtained $F(3, 1)$. When calculating the directrix, P_3 used $y = \frac{4ac-b^2-1}{4a}$ as the expression for this. However, the equation obtained was $d: y = \frac{15}{32}x$, which does not correspond to the directrix of the parabola in question. In its final construction, we noticed the mistake (**Figure 11**).

Figure 11. P_3 mistake (Source: Survey data, 2022)

Note in **Figure 11** that P_3 considered $a = 8$ and in the question we have $a = 4$. This way of calculating the focus and directrix can be found in the work of Lima (2014).

Participant P_4 entered the equation given as the function $f(x) = 4x^2 - 24x + 37$ in the input field. Then, with the "point" tool, he marked the point $A(3, 1)$, just visually as the vertex of the parabola. We understand this as an *affirmative intuition* in which P_4 acts based on what he considers right, starting from a visual perception. To calculate the focus, P_4 inserted the equation $eq1$: $16y - 17 = 0$, finding a straight line that intersects the parabola at points B and C, with the focus being the midpoint of the segment $[BC]$ (**Figure 12**).

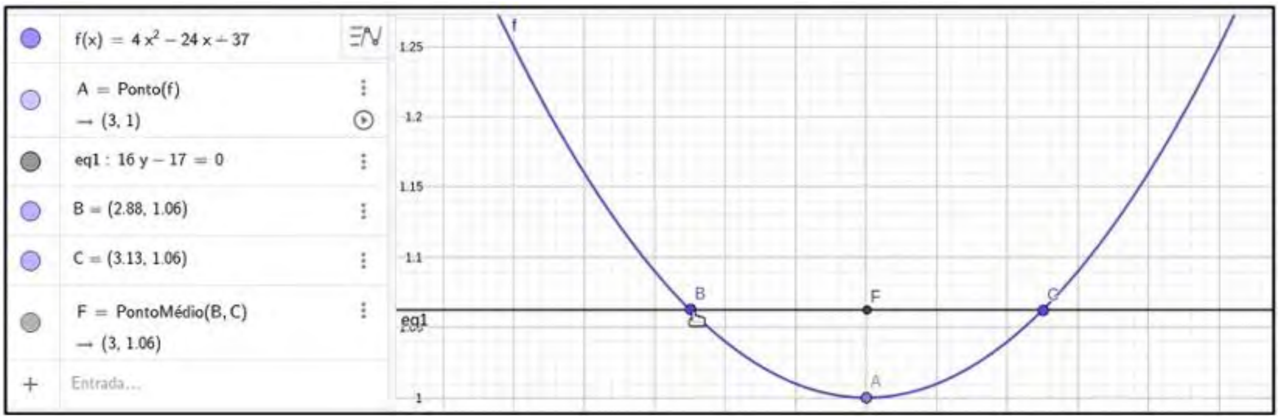

Figure 12. P_4 formulation–Part 1 (Source: Survey data, 2022)

To calculate the directrix, P_4 inserted the point $D = (3, \frac{15}{16})$ in the input field and presented **Figure 13** as the final construction:

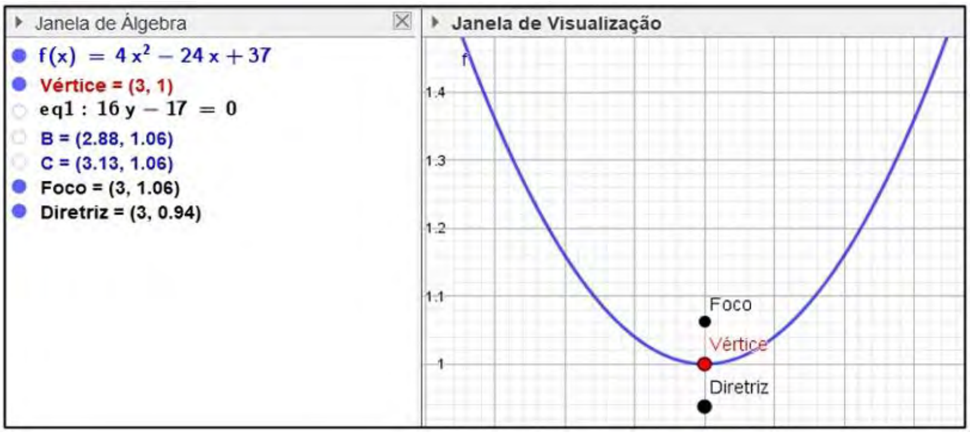

Figure 13. P_4 formulation–Part 2 (Source: Survey data, 2022)

Note that P_4 did not draw the straight line, but rather presented a point that is symmetrical to the focus and that passes through the directrix of the parabola, in accordance with the definitions pointed out in the preliminary analysis in Lima (2014) and Venturi (2003).

Participant P_5, like the others, inserted the function $f(x) = 4x^2 - 24x + 37$. However, your findings in the video record do not correspond to the actual values for the focus and directrix (**Figure 14**).

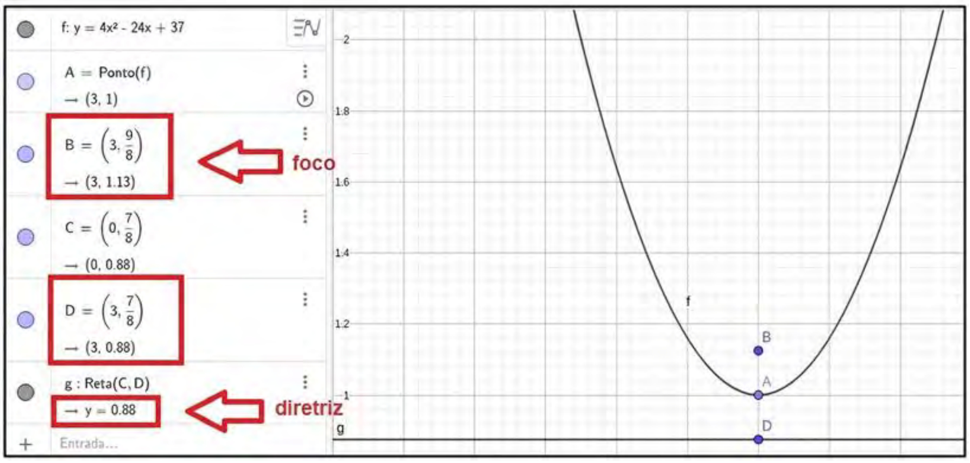

Figure 14. P_5 formulation (Source: Survey data, 2022)

In its final construction P5 presented these same values, however its written records did not allow us to identify what type of intuitions generated such conjectures.

Participant P_6 inserted the point $A(3, 1)$, the vertex of the parabola, and then inserted the equation. His path was different from the others, as he sought to construct the elements of the parabola based on knowledge of plane geometry. Through *conjectural intuitions*, given the equation and the vertex, P_6 demarcated the point $(0, 1.06)$ on the Oy axis. After that, he constructed a straight line parallel to the Ox axis passing through the vertex and a line perpendicular to the Ox axis also passing through the vertex, as we have in **Figure 15**.

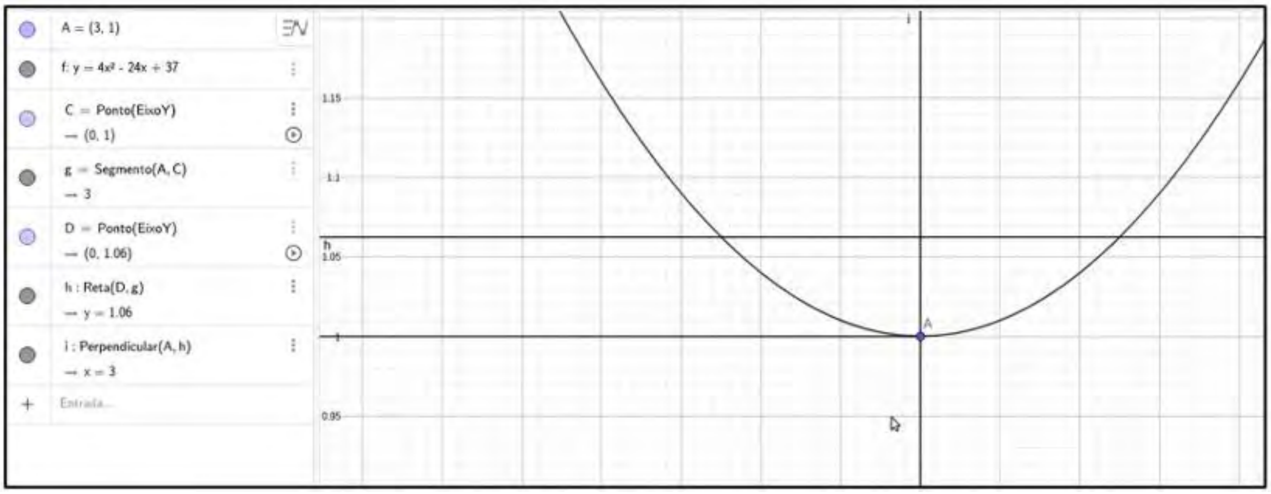

Figure 15. P_6 formulation (Source: Survey data, 2022)

P_6 hid the straight lines and the straight-line segment. Using the "reflection in relation to a point" tool, he constructed a point symmetric to the focus, which theoretically should be on the straight line. P_6, like P_4, did not construct the directrix, just a point belonging to it and symmetrical to the focus, as we can see in **Figure 16**.

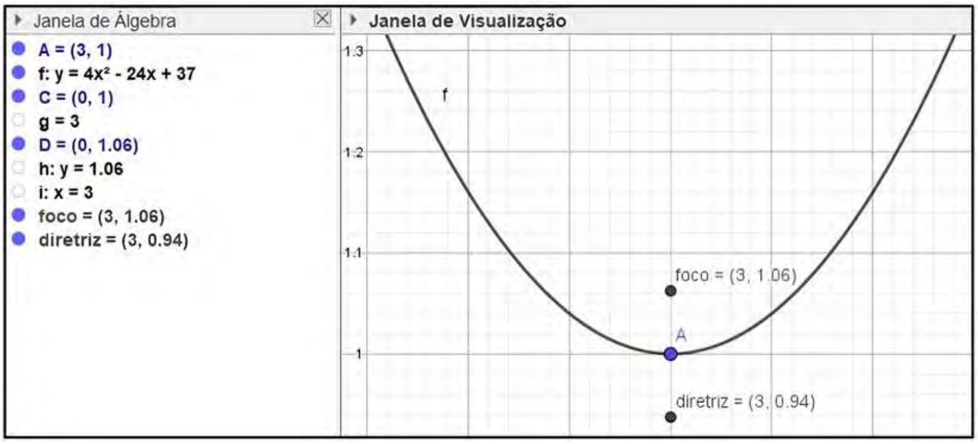

Figure 16. Final construction of P_6 (Source: Survey data, 2022)

P7's intuitions, of an *affirmative* and *conjectural* nature, were demonstrated both in his manual calculations and verbalized. The knowledge about the parabola from the perspective of analytical geometry, even after discussions, has not yet been truly grasped by P_7. His language regarding the formal terms "focus, vertex and directrix" does not yet constitute a consolidated knowledge. But we noticed in subject's behavior that there is a notion of how to proceed, and this was made possible by GeoGebra.

P_7, after reflection and dialogue with his peers, made some notes and began the construction by inserting the point $A(3, \frac{17}{16}) = (3;\ 1{,}06)$, which corresponds to a point belonging to the directrix of the parabola. Then, he inserted the points $B(3, \frac{15}{16})$ and $C(3, 1)$ into the input field, representing a point on the directrix and the vertex of the parabola. He drew a straight line parallel to the Ox axis passing through B, considering it as the straight line and used the parabola tool, obtaining $x^2 - 6x - 0.25y = -9.25$, which is the same equation $y = 4x^2 - 24x + 37$ when multiplied by $\frac{1}{4}$. This can be seen in **Figure 17**.

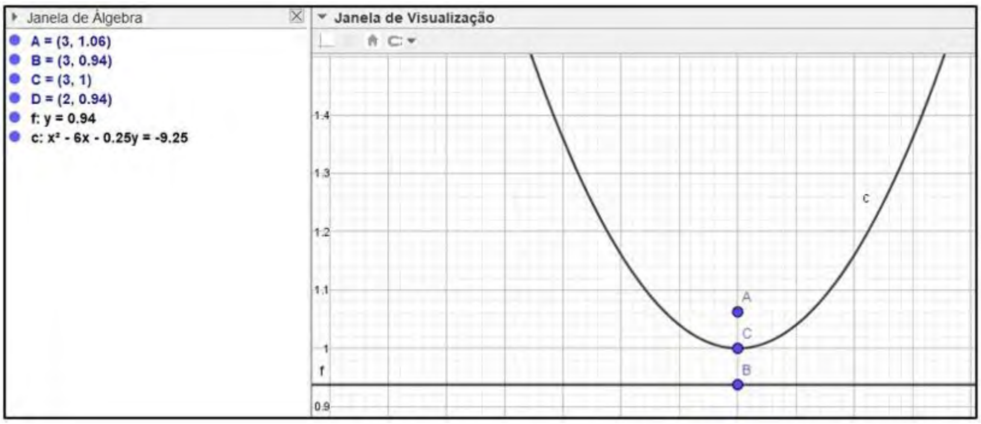

Figure 17. P_7 formulation (Source: Survey data, 2022)

Participant P_8 performed all manual calculations and inserted into the GeoGebra input field the given equation, the vertex coordinates (*A*), the focus coordinates (*B*), the directrix (*g*) and a segment that connects *A* and *B*, which is not part of the solution. His video recording was inconclusive as it was incomplete and its final construction in the software can be seen in **Figure 18**.

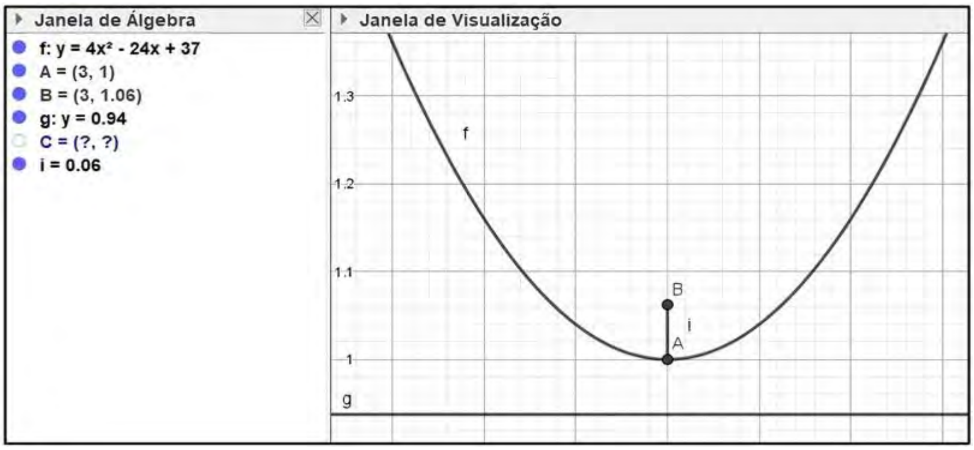

Figure 18. P_8 formulation (Source: Survey data, 2022)

Regarding the *validation situation*, the participants discussed their different solutions and strategies, as well as the ways to sketch them in the software. In particular, participants P_2, P_3 and P_7 stood out in their different strategies to solve the didactic situation, however, only P_7 was available to demonstrate it on the board to the others, as well as present an argument to validate the solution, as shown **Figure 19**.

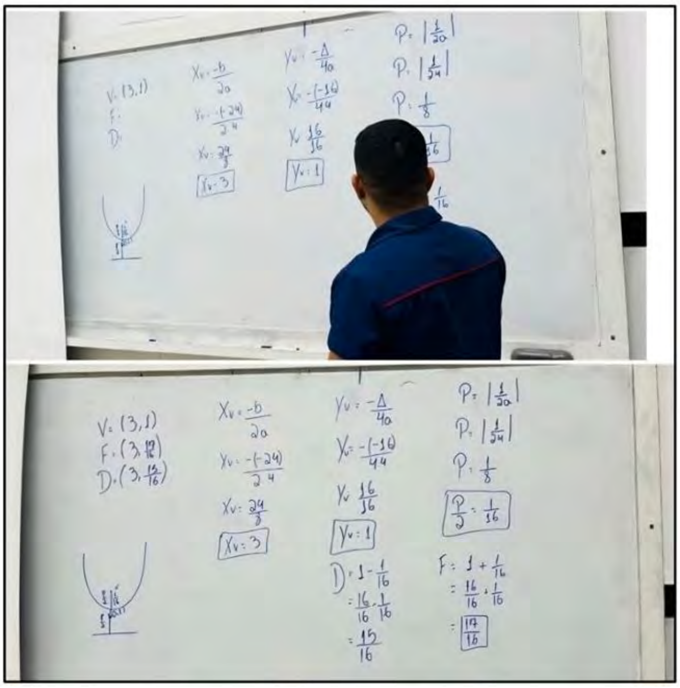

Figure 19. Validation performed by participant P_7 (Source: Survey data, 2022)

Note in **Figure 19** that P_7 calculated the vertex of the parabola, followed by the parameter *p*. In the lower left corner, we observe how his reasoning develops for calculating the directrix and focus. First, he sketched the graph, and then showed the coordinates of a point belonging to the directrix and the focus *F*. P_7 took care to present a linear, organized path, the result of a mathematical result that starts from a systematization of knowledge, in the process of being validated, which means that his path crossed the first two intuitive categories, culminating in *anticipatory intuitions* presented in the during the validation of the didactic situation. Other arguments were presented by the group and summarized by another representative of the class, in this case by P_6, as seen in **Figure 20**.

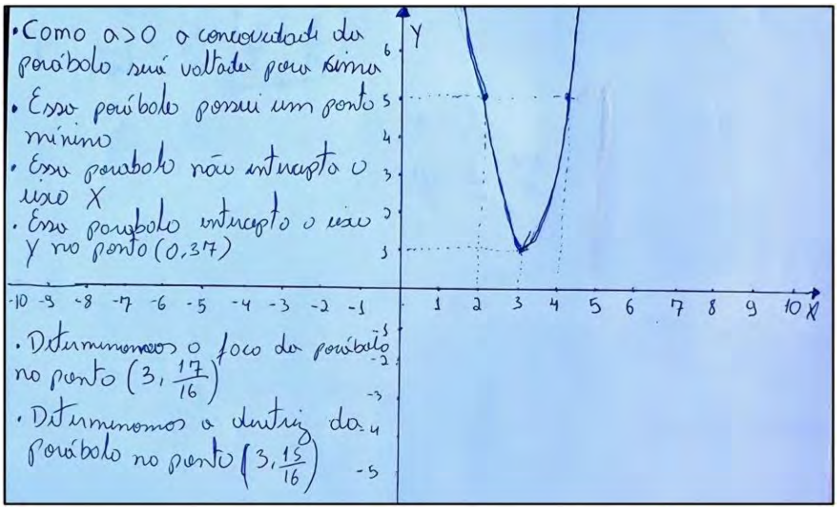

Figure 20. Validation presented by class (Source: Survey data, 2022)

Note in **Figure 20** that in the first part of the class's argument, the parabola was observed from the point of view of quadratic functions, in a natural way, which we understand as the result of *affirmative intuitions*, already mentioned previously. Only then, upon realizing that the equation did not have real and exact roots, did the participants look for other ways to construct the parabola, starting with the calculation of the vertex, moving on to the search for the focus and the directrix.

After the argument, the researcher began the institutionalization situation, based on the work of Lima (2014), which demonstrates that the equation of a parabola with vertex $V(x_0, y_0)$ and symmetry axis parallel to the y axis has the standard structure $(x - x_0)^2 = 2p(y - y_0)$. In the case of this question, the equation was presented in its explicit form $y = ax^2 + bx + c$, where to find the standard form, it is necessary to know the values x_0, y_0 and p (**Figure 21**).

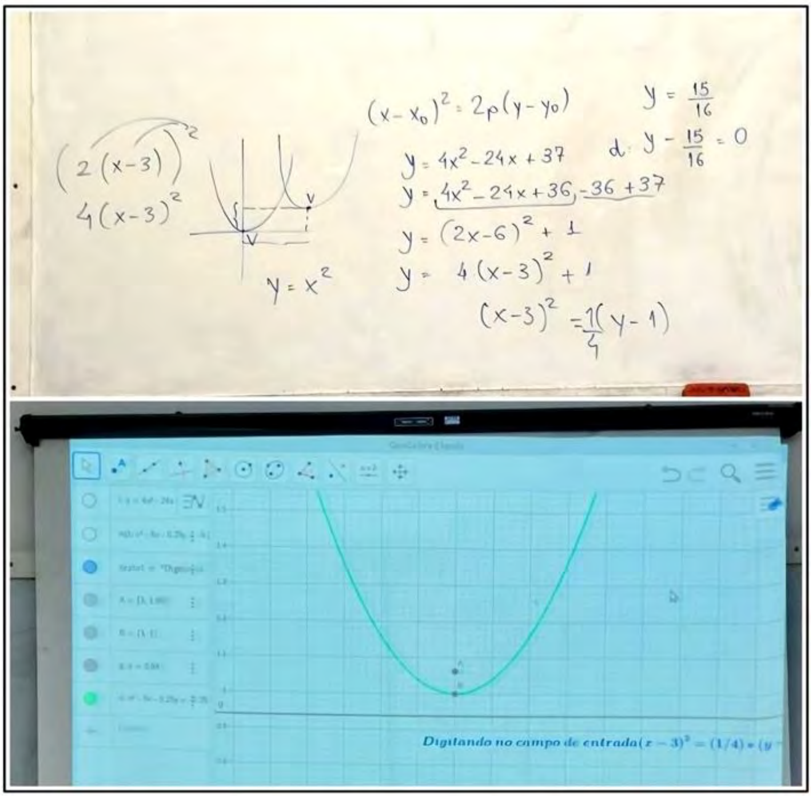

Figure 21. Record of moment of institutionalization of teaching situation (Source: Survey data, 2022)

Initially, the researcher demonstrated an alternative solution on the board and then transposed it into GeoGebra, as a way of clarifying any doubts and seeking to consolidate the participants' mathematical knowledge.

A Posteriori Analysis & Internal Validation

This didactic situation was created by the researcher, with the aim of treating the parabola equation in its explicit form, aiming for participants to recognize its main elements. In the *action situation*, participants were expected to outline the data of the question, demarcating their solution strategies in the pencil and paper environment. We established two possible strategies. The first would be that these participants would use Bhaskar's formula to search for these elements, given the association between the explicit equation of the parabola and the 2nd degree equation, thus expressing *affirmative intuitions* based on their prior knowledge and experiences. The second, in the opposite situation, would be that, when developing a reasoning path based on *affirmative* and *conjectural intuitions*, they would transform the explicit equation of the parabola into a reduced equation and recognize its elements (standard or ideal solution). In fact, the second strategy occurred more recurrently. Pre-service teachers also have an ingrained view of the parabola linked to the graph of the quadratic function, which can be seen in both the audio and written records of the research, in addition to being explained in our preliminary analysis.

Furthermore, intrinsic certainty, as proposed by the work of Fischbein (1987), configures the record of an *affirmative intuition*, with regard to the attempt to immediately calculate the roots of the equation (participants P_1, P_5, and P_8), or consider the coefficients *a*, *b* and *c* as the main elements of this curve (participants P_6, P_7, and P_8). Only participants P_1 and P_3 presented models of *conjectural intuitions* mentioning, respectively, the search for the focus and the calculation of the derivative of the equation to find the vertex of the parabola.

Regarding the formulation situation, we aimed for participants to rewrite the given equation as a perfect square trinomial. If this stage were started in GeoGebra, it was expected that participants would express intuitions of a *conjectural* and *anticipatory* nature, based on viewing the sketched graph. However, we noticed relatively different paths at this stage of the didactic situation. Among our observations, we can highlight:

1. Only participants P_2 and P_3 directly used knowledge in analytical geometry, demonstrating *conjectural intuitions* about the topic.
2. P_2 was the only one to use sequenced and pre-established commands in GeoGebra and to develop *anticipatory* and *conclusive intuitions*.
3. The other participants began their procedures by calculating the roots, vertex, then calculating focus and directrix, in this order.
4. P_6, as he did not know some GeoGebra resources, used his knowledge in plane geometry within the construction when developing his solution.
5. P_7 raised several relevant questions, standing out with regard to participation and intuitive demonstrations.

In the validation situation, participants were expected to find the correct values for the focus, vertex, directrix, and parameter *p* and present their findings with the help of the GeoGebra software. In this didactic situation, in particular, validation was carried out collectively and one of the participants (P_7) was designated by the class to present the point of view of those present. At this stage we notice the mark of *affirmative* and *conjectural intuitions*, presented by P_7 when characterizing the given equation considering it as a quadratic function, followed by procedures for mathematicians to calculate the focus, the vertex and the directrix. We also noticed the manifestation of *anticipatory intuitions* during this characterization in which P_7 described the synthesis of the route through deductive reasoning.

Finally, in the institutionalization situation, the researcher reviewed the arguments presented and demonstrated the concept of the explicit equation of the parabola and the relationship between the equation of the parabola in explicit form and the structure of a quadratic function. To this end, the researcher was supported by the work of Lima (2014), carrying out this demonstration succinctly on the board and transposing it into GeoGebra.

The main difficulties in the study of the parabola often arise from the transition between graphical and algebraic representations. Understanding the relationship between the elements of the parabola and its algebraic equation in different forms, as well as relating this algebraic representation to the graphical interface, can be challenging for students, requiring abstract thinking skills that we aim to develop throughout their education.

De Sousa et al. (2023) emphasize the importance of pedagogical approaches that integrate graphical visualizations and mathematical modeling in teaching the parabola. These strategies provide a deeper and more connected understanding of the concept, facilitating the overcoming of difficulties.

The study of the parabola is not only relevant to teacher training but also to mathematics Education in general. Well-prepared teachers to teach concepts related to the parabola have the potential to spark students' interest, providing a solid foundation for understanding more advanced concepts in algebra and geometry.

The parabola is a crucial topic in mathematics teacher training due to its significance in various mathematical areas and its practical applicability. Recognizing common difficulties in this subject and adopting effective pedagogical approaches are essential to ensure that future teachers can effectively convey this knowledge, thereby contributing to the improvement of mathematics education.

CONCLUSIONS

This work started from a problematization, observed in a bibliographical survey, of some gaps that permeate the initial training of mathematics teachers in Brazil, regarding the teaching of parabolas. The recurrent model of traditional approach and the fragmentation of the subject, disconnected from reality and related topics, such as quadratic functions, are an example of these pre-existing gaps found. In this way, we developed a DE to verify the possible didactic obstacles that cause difficulties in the way in which the student in initial training understands and, in a future vision, would approach the parabola in his locus of work.

In the preliminary analysis we demarcated some epistemological and didactic aspects in the teaching of parabolas and aspects of TDS and the categories of intuitive reasoning, which structured this teaching session, while in the a priori analysis we structured the didactic situation and developed it in the phase of experimentation. In our a posteriori analysis, we highlight the importance of working on the parabola within the scope of geometry during the degree, approaching it mainly from the concept of geometric place, with a view to contributing to the future teacher developing a perspective that allows him to construct a relationship between the geometric and analytical vision of the parabola. The intuitive categories allowed us to interpret how knowledge about the parabola has been consolidated in this group of participants. When observing the discussions, it is noted that the parabola was naturally associated with the graph of the quadratic function and, in the background, given the need for the didactic situation, this was considered a conic curve, with its particularities.

The study of the parabola is central in the training of mathematics teachers, as it enables them to develop a deep understanding of algebraic and geometric concepts and their interrelation. This understanding is crucial for the future teachers' ability to convey these concepts clearly and meaningfully to students. The parabola is often encountered in practical contexts, such as projectile trajectories, resource optimization, and modeling natural phenomena. The ability to apply knowledge about parabolas is valuable for future teachers as it empowers them to address real-world problems in formal mathematical language, connecting students' everyday experiences with mathematics.

In this sense, we recommend an approach supported by technology, such as GeoGebra, so that these aspects (geometric and analytical) and their relationships can be worked on more proficiently, through the lens of dynamic geometry, aiming to contribute to the training of future mathematics teachers.

Author contributions: RTS: manuscript writing & data collection & **FRVA & HMBC:** orientation & revision of final text. All authors have agreed with the results and conclusions.

Funding: The first two authors are grateful for the encouragement and financial contribution of the National Council for Scientific and Technological Development (CNPq) for the development of this research in Brazil. The third author is grateful for the funding provided by National Funds through FCT–Foundation for Science and Technology, I. P., under the projects (https://doi.org/10.54499/UIDB/00194/2020) and UIDP/00194/2020 (https://doi.org/10.54499/UIDP/00194/2020).

Ethical statement: The authors stated that the research complies with the ethical parameters adopted at the Federal Institute of Education, Science and Technology of Ceará, which consider the regulations of the National Research Ethics Committee, based on Resolution 466/12, Resolution 510/2016, and the principles contained in the Codes of Ethics of the Committee on Publication Ethics, according to legislation available at: https://bvsms.saude.gov.br/bvs/saudelegis/cns/2016/res0510_07_04_2016.html valid throughout Brazil

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Almouloud, S. A., & da Silva, M. J. F. (2012). Engenharia didática: Evolução e diversidade [Didactic engineering: Evolution and diversity]. *Revemat: Revista Eletrônica de Educação Matemática* [*Electronic Magazine of Mathematics Education*], *7*(2), 22-52. https://doi.org/10.5007/1981-1322.2012v7n2p22

Alves, F. R. V. (2017). Didática das ciências e matemática (DCEM): Surgimento e implicações para a formação do professor [Didactics of science and mathematics (DCEM): Emergence and implications for teacher training]. *Investigações em Ensino de Ciências* [*Research in Science Teaching*], *22*(3), 291-320. https://doi.org/10.22600/1518-8795.ienci2017v22n3p291

Artigue, M. (2020). Méthodologies de recherche en didactique des mathématiques: Où en sommes-nous? [Research methodologies in mathematics education: Where are we?]. *Revista Educação Matemática e Pesquisa* [*Mathematics Education and Research Journal*], *22*(4), 25-64. https://doi.org/10.23925/1983-3156.2020v22i3p025-064

Bohrer, A., & da Silva Tinti, D. (2021). Mapeamento de pesquisas sobre a função quadrática em contextos de ensino e/ou aprendizagem da matemática [Mapping research on the quadratic function in mathematics teaching and/or learning contexts]. *Educação Matemática Pesquisa* [*Mathematics Education Research*], *23*(1), 201-230. https://doi.org/10.23925/19833156.2021v23i1p201-230

Brousseau, G. (2002). *Theory of didactical situations in mathematics: Didactique des mathématiques* [*Didactics of mathematics*]. Kluwer Academic Publishers.

Brousseau, G., & Gibel, P. (2005). Didactical handling of students' reasoning processes in problems solving situations. In C. Laborde, M. J. Perrín-Glorian, & A. Sierpinska (Eds.), *Beyond the apparent banality of the mathematics classroom* (pp. 13-58). Springer. https://doi.org/10.1007/0-387-30451-7_2

Cerqueira, A. A. (2015). *Parábola e suas aplicações* [*Parabola and its applications*] [Master's thesis, Universidade Federal da Bahia].

De Sousa, R. T., Alves, F. R. V., & Aires, A. P. (2023). Categories of intuitive reasoning in the teaching of parabolas: A structured practice in didactic engineering. *International Electronic Journal of Mathematics Education, 18*(4), em0746. https://doi.org/10.29333/iejme/13514

De Sousa, R. T., Alves, F. R. V., & Souza, M. J. A. (2022). Aspectos da parábola e da catenária: Um estudo à luz da geometria dinâmica [Aspects of the parabola and the catenary: A study in the light of dynamic geometry]. *Revemat: Revista Eletrônica de Educação Matemática* [*Electronic Journal of Mathematics Education*]*, 17*, 1-22. https://doi.org/10.5007/1981-1322.2022.e88156

Feltes, C. M., & Puhl, C. S. (2016). Gráfico da função quadrática: Uma proposta de ensino potencialmente significative [Graph of the quadratic function: A potentially significant teaching proposal]. *Scientia cum Industria* [*Science with Industry*]*, 4*(4), 202-206. https://doi.org/10.18226/23185279.v4iss4p202

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Mathematics Educational Library.

Leonardo, F. M. (2016). *Conexões com a matemática* [*Connections to mathematics*]. Moderna.

Lima, E. L. (2001). *Exame de textos: Análise de livros de matemática para o ensino médio* [*Text exam: Analysis of mathematics books for high school*]. SBM/IMPA.

Lima, E. L. (2014). *Geometria analítica e algebra linear* [*Analytic geometry and linear algebra*]. IMPA.

Lucena, R., & Gitirana, V. (2016). Articulações internas à matemática: A parábola e a função quadrática com o GeoGebra [Internal articulations in mathematics: The parabola and the quadratic function with GeoGebra]. *Educação Matemática em Revista* [*Mathematics Education in Journal*]*, 51*, 25-34.

Macedo, H. R. (2015). *Estudo sistemático das parabolas* [*Systematic study of the parabolas*] [Master's thesis, Universidade Federal da Paraíba].

Siqueira, C. A. F. (2016). *Um estudo didático das cônicas: Quadros, registros e pontos de vista* [*A didactic study of conics: Tables, records and points of view*] [Master's thesis, Pontifícia Universidade Católica de São Paulo].

Venturi, J. J. (2003). *Cônicas e quádricas* [*Conics and quadrics*]. Livrarias Curitiba.